# General Relativistic Anisotropic Hydrostatic Equilibrium Equations for a Magnetized Compact Object

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# Introduction

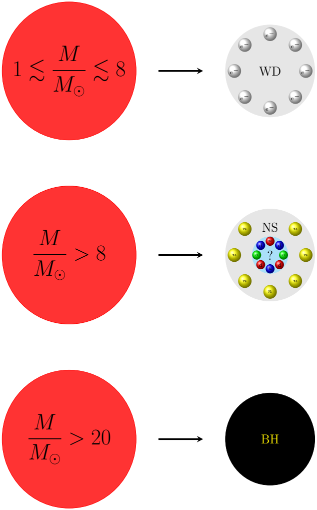



Figure: Diagram of the final state of stars.

Figure: Sirius A and B (Hubble telescope).

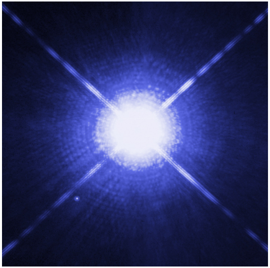
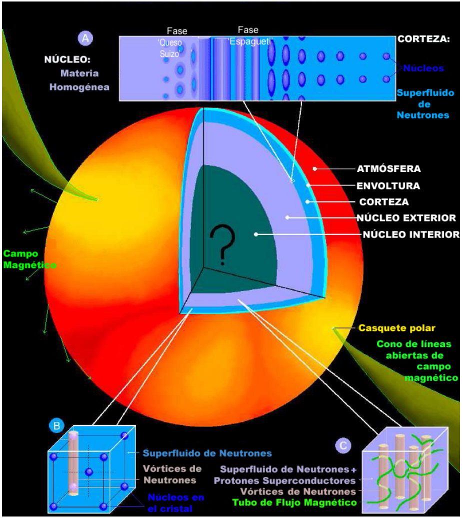


Figure: Crab nebula (Hubble and Chandra).

# Introduction
## Neutron Stars. Quark Stars?

- $\mathrm{M} \sim 1.4\mathrm{M}_{\odot}$
- $\mathrm{R} \sim 12\,\mathrm{km}$

- $\rho \sim 10^{14}\,\mathrm{g/cm^3}$
- $\mathcal{B} \sim 10^{13} - 10^{15}\,\mathrm{G}$



Measurements of the masses or radii of CO can strongly constrain the neutron star matter equation of state and rule out theoretical models of their composition.

The observed range of neutron star masses, however, has hitherto been too narrow to rule out many predictions of exotic non-nucleonic components.

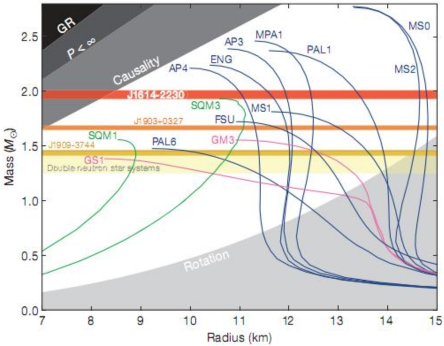



Figure: Mass–Radius diagram[†].

[†] P. Demorest, T. Pennucci, S. Ransom, M. Roberts and J. Hessels, Nature **467**, 1081 (2010)

# TOV equation

To found the static structure of a relativistic spherical star we most solve the Einstein's equation

$$G^{\mu}{}_{\nu} \equiv R^{\mu}{}_{\nu} - \frac{1}{2} R g^{\mu}{}_{\nu} = 8\pi G \mathcal{T}^{\mu}{}_{\nu}, \tag{1}$$

$(\mu, \nu = 0, 1, 2, 3)$, using the Schwarzschild metric

$$ds^2 = -e^{2\nu} dt^2 + e^{2\lambda} dr^2 + r^2 d\Omega^2, \quad d\Omega^2 = d\theta^2 + \sin^2\phi d\phi^2, \tag{2}$$

and the energy momentum tensor

$$\mathcal{T}^{\mu}{}_{\nu} = (\epsilon + P) u^{\mu} u_{\nu} + P g^{\mu}{}_{\nu}, \tag{3}$$

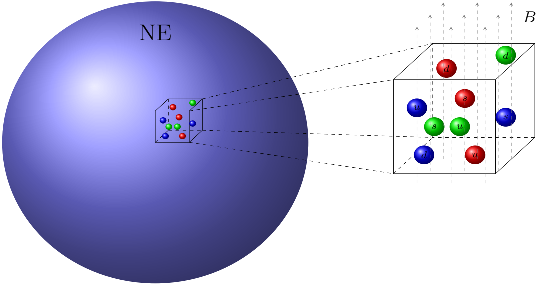
we obtain the TOV equations

$$\begin{aligned} \frac{dM}{dr} &= 4\pi G \epsilon, \\ \frac{dP}{dr} &= -G \frac{(\epsilon + P)(M + 4\pi P r^3)}{r^2 - 2rM}, \end{aligned}$$

with boundary conditions $P(R) = 0$, $M(0) = 0$.

# Pressures anisotropy due to a magnetic field

A magnetic field breaks the spherical symmetry and produce an anisotropy in the pressures inside the object.



# Pressures anisotropy due to a magnetic field

## Magnetized Fermion Gas

The thermodynamical potential in the framework of the MIT Bag Model is given by:

$$\Omega_{i,v}(\mu_i, T) = -\frac{d_i T}{(2\pi)^3}\int_p [\ln[1+\exp\{-\frac{E_{p,i}^{n,\eta}-\mu_i}{T}\}]d^3p],$$

where,

$$E_{p,i}^{n,\eta} = \sqrt{p_z^2+p_\perp^2+m_i^2}, \;\; p_\perp^2 = m_i^2\{(\sqrt{\frac{\mathcal{B}}{\mathcal{B}_i^c}(2n+1-\eta)+1}-\eta y_i\mathcal{B})^2-1\},$$

$i = e,\, u,\, d,\, s$, $B_i^c = m_i^2/|e_i|$ is the critical magnetic field, $y_i = |Q_i|/m_i$ accounts for the AMM and $\eta = \pm 1$ correspond to the orientations of the particle magnetic moment, parallel or antiparallel to the magnetic field. Parameters $d_e = 1$ and $d_{u,d,s} = 3$ are degeneracy factors.

# Pressures anisotropy due to a magnetic field

## Magnetized Fermion Gas

For degenerate magnetized strange quark matter, the number density, energy density and pressures are given by the expressions:

$$N = -\sum_i \frac{\partial \Omega_{i,v}}{\partial \mu_i} = \sum_i N_i^0 \frac{B}{B_i^c} \sum_n^{n_{max}^i} \sum_{\eta=\pm 1} p_{F,i}^{\eta} \,,$$

$$\varepsilon = \Omega + \mu N = B \sum_i \mathcal{M}_i^0 \sum_n^{n_{max}^i} \sum_{\eta=\pm 1} \left( x_i p_{F,i}^{\eta} + h_i^{\eta\,2} \ln \frac{x_i + p_{F,i}^{\eta}}{h_i^{\eta}} \right) ,$$

$$P_{\parallel} = -\Omega = B \sum_i \mathcal{M}_i^0 \sum_n^{n_{max}^i} \sum_{\eta=\pm 1} \left( x_i p_{F,i}^{\eta} - h_i^{\eta\,2} \ln \frac{x_i + p_{F,i}^{\eta}}{h_i^{\eta}} \right) ,$$

$$P_{\perp} = -\Omega - \mathcal{M} B = B \sum_i \mathcal{M}_i^0 \sum_n^{n_{max}^i} \sum_{\eta=\pm 1} \left( 2 h_i^{\eta} \gamma_i^{\eta} \ln \frac{x_i + p_{F,i}^{\eta}}{h_i^{\eta}} \right) ,$$

The sum over the Landau levels $n$ is up to $n_{max}^i = I\left[\left((x_i + \eta y_i B)^2 - 1\right)\, B_i^c/(2B)\right]$,where $I[z]$ denotes the integer part of $z$.

# Pressures anisotropy due to a magnetic field

## Magnetized Fermion Gas

We have defined the dimensionless quantities:

$$
\begin{aligned}
\mathcal{M}_i^0 &= \frac{e_i d_i m_i^2}{4\pi^2}, \quad N_i^0 = \frac{d_i m_i^3}{2\pi^2}, \quad x_i = \mu_i/m_i \\
p_{F,i}^{\eta} &= \sqrt{x_i^2 - h_i^{\eta\,2}}, \quad h_i^{\eta} = \sqrt{\frac{B}{B_i^c}\,(2n+1-\eta)+1} - \eta y_i B \\
\gamma_i^{\eta} &= \frac{B\,(2n+1-\eta)}{2B_i^c\sqrt{(2n+1-\eta)B/B_i^c+1}} - \eta y_i B,
\end{aligned}
$$

where $x_i$ is the dimensionless chemical potential, $p_{F,i}$ corresponds to the modified Fermi momentum due to the magnetic field and $h_i^{\eta}$ corresponds to the magnetic mass.

# Pressures anisotropy due to a magnetic field

## MIT Bag model for SQM

In order to study the matter inside the star we use the MIT Bag model.The stelar equilibrium conditions are then determined by the following relations:

$$d \leftrightarrow u + e^- + \bar{\nu}_e \quad s \leftrightarrow u + e^- + \bar{\nu}_e \quad u + s \leftrightarrow d + u$$

So that we have to solve the system of equations

$$\begin{aligned} \mu_u + \mu_e - \mu_d = 0, \quad \mu_d - \mu_s &= 0 \quad \beta \text{ equilibrium} \\ 2n_u - n_d - n_s - 3n_e &= 0 \quad \text{charge neutrality} \\ n_u + n_d + n_s - 3n_B &= 0 \quad \text{baryon number conservation.} \end{aligned}$$

# Pressures anisotropy due to a magnetic field

## MIT Bag model for SQM

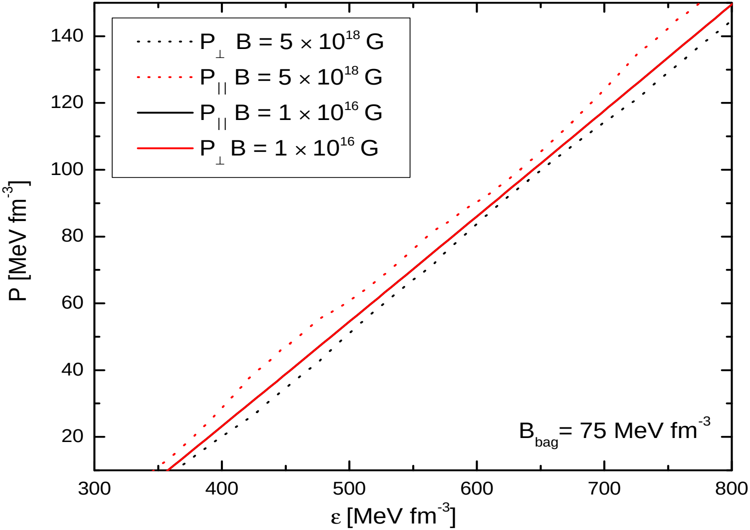



Figure: Equations of state for magnetized strange quark matter.

# Pressures anisotropy due to a magnetic field

## MIT Bag model for SQM

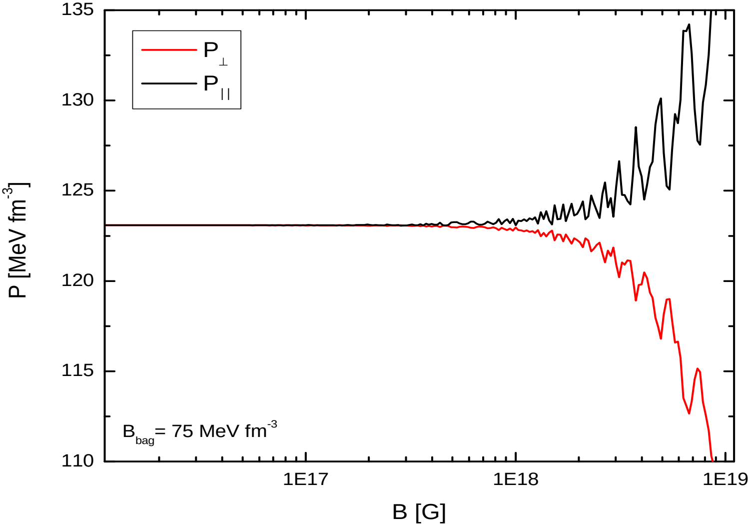



Figure: Dependence of the pressures whit respect to the magnetic field.

# Pressures anisotropy due to a magnetic field

## MIT Bag model for SQM

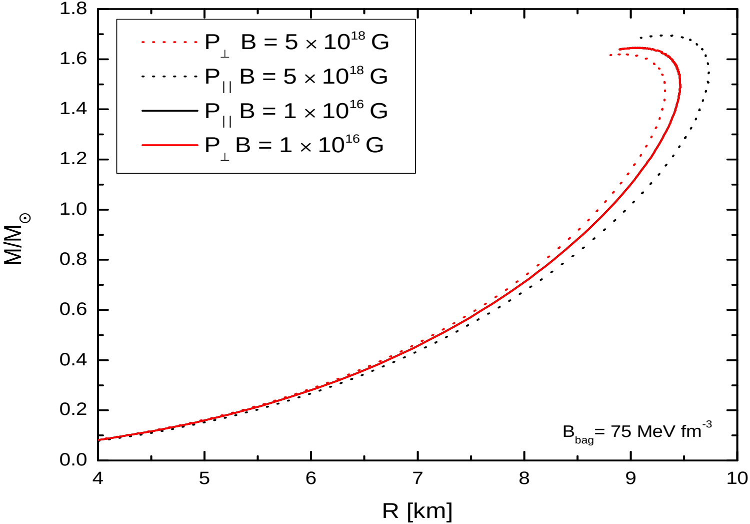



Figure: Mass-Radius diagram comparing the effects of taking parallel or perpendicular pressure.

# Cylindrical symmetry[†]

For the study of the impact of pressures anisotropies in the structure of magnetized stars we will use the cylindrically symmetric metric

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2 d\phi^2 + e^{2\Psi}dz^2 \tag{5}$$

where $\Phi$, $\Lambda$, $\Omega$, and $\Psi$ are functions of $r$ only.
With this metric we have for the nonzero Einstein tensor components

$$\begin{aligned}
G_t^t &= e^{-2\Lambda}(\Psi'' + \Psi'^2 - \Psi'\Lambda' - \frac{1}{r}\Lambda' + \frac{1}{r}\Psi') \\
G_r^r &= e^{-2\Lambda}(\Psi'\Phi' + \frac{1}{r}\Phi' + \frac{1}{r}\Psi') \\
G_\phi^\phi &= e^{-2\Lambda}(\Phi'' + \Phi'^2 - \Phi'\Lambda' + \Psi'' + \Psi'^2 - \Psi'\Lambda' + \Psi'\Phi') \\
G_z^z &= e^{-2\Lambda}(\Phi'' + \Phi'^2 - \Phi'\Lambda' - \frac{1}{r}\Lambda' + \frac{1}{r}\Phi')
\end{aligned}$$

[†] C. S. Trendafilova and S. A. Fulling, "Static solutions of Einstein's equations with cylindrical symmetry," Eur. J. Phys. **32**, 1663 (2011) [arXiv:1101.4668 [gr-qc]].

# Cylindrical symmetry[†]

Energy momentum tensor for magnetized mater is[†]

$$\mathcal{T}^{\mu}{}_{\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P_{\perp} & 0 & 0 \\ 0 & 0 & P_{\perp} & 0 \\ 0 & 0 & 0 & P_{\parallel} \end{pmatrix}. \tag{6}$$

From the Einstein field equations in natural units we get the following four differential equations:

$$\begin{aligned}
4\pi\epsilon &= -e^{-2\Lambda}(\Psi'' + \Psi'^2 - \Psi'\Lambda' - \frac{1}{r}\Lambda' + \frac{1}{r}\Psi') \\
4\pi P_{\perp} &= e^{-2\Lambda}(\Psi'\Phi' + \frac{1}{r}\Phi' + \frac{1}{r}\Psi') \\
4\pi P_{\perp} &= e^{-2\Lambda}(\Phi'' + \Phi'^2 - \Phi'\Lambda' + \Psi'' + \Psi'^2 - \Psi'\Lambda' + \Psi'\Phi') \\
4\pi P_{\parallel} &= e^{-2\Lambda}(\Phi'' + \Phi'^2 - \Phi'\Lambda' - \frac{1}{r}\Lambda' + \frac{1}{r}\Phi')
\end{aligned}$$

[†] R. G. Felipe, H. J. Mosquera Cuesta, A. Perez Martinez and H. Perez Rojas, “Quantum instability of magnetized stellar objects,” Chin. J. Astron. Astrophys. **5**, 399 (2005) [astro-ph/0207150].

# Hydrostatic Equilibrium Equations for a Magnetized CO

By doing some algebra with the previous system of equation we get

$$\begin{aligned}
P'_\perp &= -\Phi'(\epsilon + P_\perp) - \Psi'(P_\perp - P_\parallel) \\
4\pi e^{2\Lambda}(\epsilon + P_\parallel + 2P_\perp) &= \Phi'' + \Phi'(\Psi' + \Phi' - \Lambda') + \frac{\Phi'}{r} \\
4\pi e^{2\Lambda}(\epsilon + P_\parallel - 2P_\perp) &= -\Psi'' - \Psi'(\Psi' + \Phi' - \Lambda') - \frac{\Psi'}{r} \\
4\pi e^{2\Lambda}(P_\parallel - \epsilon) &= \frac{1}{r}(\Psi' + \Phi' - \Lambda')
\end{aligned}$$

This is a system of differential equations in the variables

$$P_\perp, \ \ \Phi, \ \ \Lambda, \ \ \Psi, \tag{7}$$

provided the equations of state

$$\epsilon \to f(P_\perp), \ \ P_\parallel \to f(\epsilon) \tag{8}$$

and initial conditions.

## Initial conditions

Since the differential equations involve factors of $1/r$ we will make power series expansions of $p$, $\Phi$, $\Psi$, and $\Lambda$ around $r = 0$ to find initial condition suitable for numeric calculation

$$P_\perp = P_{\perp 0} + P_{\perp 1} r, \quad (9)$$

$$\Lambda = \Lambda_0 + \Lambda_1 r, \quad (10)$$

$$\Phi = \Phi_0 + \Phi_1 r + \Phi_2 r^2, \quad (11)$$

$$\Psi = \Psi_0 + \Psi_1 r + \Psi_2 r^2. \quad (12)$$

We take also

- $\Psi = \Phi = \Lambda = 0$ at $r = 0$ so that the corresponding metric coefficients are equal to $1$ at that point.
- $\Psi' = \Phi' = 0$ to get smooth solutions at the axis.

# Initial conditions

By substitution of these conditions in the system of differential equations we find

$$
\begin{aligned}
P_\perp(0) &= P_{\perp 0} \\
\Lambda(0) &= 0 \\
\Phi(0) &= \frac{1}{2}(P_{\| 0} + 2P_{\perp 0} + \epsilon_0)(r_0^2 - 2r_0) \\
\Psi(0) &= \frac{1}{2}(-P_{\| 0} + 2P_{\perp 0} - \epsilon_0)(r_0^2 - 2r_0) \\
\Phi'(0) &= 0 \\
\Psi'(0) &= 0
\end{aligned}
$$

And we also take

$$P_\|(R_\|) = 0, \quad P_\perp(R_\perp) = 0$$

For computing the mass we will use Tolman† generalization for the the mass of a source

$$m_T = \int \sqrt{-g}(T_0^0 - T_1^1 - T_2^2 - T_3^3)dV \quad (13)$$

for our cylindric metric we have

$$m_T = \int re^{\Phi+\Psi+\Lambda}(\epsilon - 2P_\perp - P_\parallel)dV \quad (14)$$

† R. C. Tolman, Relativity, Thermodynamics and Cosmology. Oxford University Press, London (1934)

# Hydrostatic Equilibrium Equations for a Magnetized CO
## Numerical solution

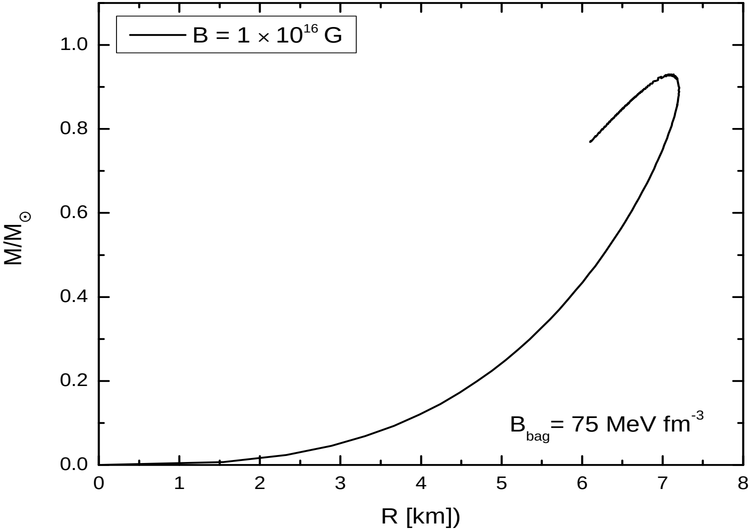



Figure: Mass-Radius diagram for $B = 10^{16}$ G.

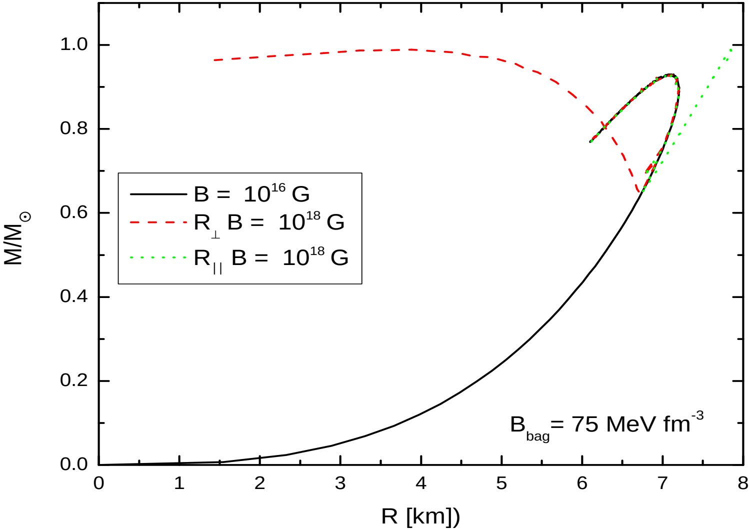



Figure: Mass-Radius diagram for $B = 10^{16}$ G and $B = 10^{18}$.

# Conclusions

1. Taking into account pressure anisotropy due to a magnetic field leads to smaller maximum masses.

2. For high values of the magnetic field the behaviour of the mass radius relation changes.

Future work

1. Make more numerical experiments varying parameters such as the Bag in order to see if there are any configurations in which the system could reach higher values of the mass.

2. Investigate other models for anisotropic equations of state.

# GRACIAS!

# OBRIGADO!

# THANKS YOU!